Topic 9 -
Variation of parameters

In this topic we will give another method to find a particular solution to a linear 2nd order ODE.

This method will work in situations where undetermined coefficients doesnt such as finding a particular solution to

$$y'' + y = \tan(x)$$

which we wouldn't be able to solve with undetermined coefficients.

The method will also work when one has non-constant coefficients.

Derivation of variation of parameters formula

Suppose we have

$$y'' + a_1(x)y' + a_2(x) = b(x)$$

where $a_1(x), a_2(x), b(x)$ are continuous on some interval I.

If you had $a_2(x)$ in front of y'' and it was never 0 you can divide by $a_2(x)$ first to put in this form

We want to find a particular solution y_p to the above equation on I.

Suppose we have two linearly independent solutions y_1 and y_2 to the homogeneous equation.

That is, assume y_1 and y_2 solve

$$y'' + a_1(x)y' + a_2(x) = 0$$

on I.

There is a theorem that says that this will imply that the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

see end of these notes for a proof

will be non-zero on I.

Let

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1 and v_2 are unknown functions to be determined. We hope to find v_1 and v_2 that make y_p a particular solution to $y'' + a_1(x)y' + a_0(x)y = b(x)$.

Note that this implies that

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$
$$= (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2)$$

We will now require that $v_1' y_1 + v_2' y_2 = 0$ to simplify our calculations. Even though this may seem restrictive, it actually ends up working.

Then,

$$y_p' = v_1 y_1' + v_2 y_2'$$

and

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plugging this into $y'' + a_1(x) y' + a_0(x) y = b(x)$

We will need

$$\left(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''\right) + a_1(x)\left(v_1 y_1' + v_2 y_2'\right) + a_0(x)\left(v_1 y_1 + v_2 y_2\right) = b(x)$$

Which becomes

$$v_1\Big(\underbrace{y_1'' + a_1(x) y_1' + a_0(x) y_1}_{0}\Big) + v_2\Big(\underbrace{y_2'' + a_1(x) y_2' + a_0(x) y_2}_{0}\Big) + v_1' y_1' + v_2' y_2' = b(x)$$

So we will need $v_1' y_1' + v_2' y_2' = b(x)$.

Thus we have two equations to solve:

$$\boxed{\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= b(x) \end{aligned}} \quad \begin{aligned} &1 \\ &2 \end{aligned}$$

two equations and two unknowns v_1', v_2'

To solve the above system, calculate

$y_1' * ① - y_1 * ②$ to get

$$(v_1' y_1 y_1' + v_2' y_2 y_1') - (v_1' y_1' y_1 + v_2' y_2' y_1) = -y_1 \cdot b(x)$$

This gives

$$v_2' \underbrace{(y_2 y_1' - y_2' y_1)}_{-W(y_1, y_2) = -(y_1 y_2' - y_1' y_2)} = -y_1 \cdot b(x)$$

So,

$$\boxed{v_2' = \frac{y_1 \cdot b(x)}{W(y_1, y_2)}}$$

Similarly, $y_2' * ① - y_2 * ②$ would give

$$(v_1' y_1 y_2' + v_2' y_2 y_2') - (v_1' y_1' y_2 + v_2' y_2' y_2) = -y_2 \cdot b(x)$$

Thus,

$$v_1' \underbrace{(y_1 y_2' - y_1' y_2)}_{W(y_1, y_2)} = -y_2 \cdot b(x)$$

So,

$$\boxed{v_1' = \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}}$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\,dx \quad \text{and} \quad v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}\,dx$$

and

$$y_p = v_1 y_1 + v_2 y_2$$

Summary

Let $a_1(x), a_0(x), b(x)$ be continuous on I.
Let y_1, y_2 be linearly independent solutions to the homogeneous equation

$$y'' + a_1(x)\,y' + a_0(x)\,y = 0.$$

Then a particular solution to

$$y'' + a_1(x)\,y' + a_0(x)\,y = b(x)$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\,dx \quad \text{and} \quad v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}\,dx$$

Ex: Let's solve

Using variation of parameters.

Step 1: To solve the homogeneous equation

$y'' - 4y' + 4y = 0$ we have the characteristic equation

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

So the homogeneous solution is

$$y_h = c_1 \underbrace{e^{2x}}_{y_1} + c_2 \underbrace{x e^{2x}}_{y_2}.$$

Step 2: Now we use variation of parameters with $y_1 = e^{2x}$ and $y_2 = xe^{2x}$.

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{2x}\left(e^{2x} + 2x e^{2x}\right) - 2x e^{2x} e^{2x}$$

$$= e^{4x} + 2x e^{4x} - 2x e^{4x}$$

$$= e^{4x}$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\,dx = \int \frac{-xe^{2x} \cdot (x+1)\,e^{2x}}{e^{4x}}\,dx$$

$$= \int (-x^2 - x)\,dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}\,dx = \int \frac{e^{2x} \cdot (x+1)\,e^{2x}}{e^{4x}}\,dx$$

$$= \int (x+1)\,dx = \frac{x^2}{2} + x$$

So,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(-\frac{x^3}{3} - \frac{x^2}{2}\right) e^{2x} + \left(\frac{x^2}{2} + x\right) x e^{2x}$$

$$= \left(\frac{x^3}{6} + \frac{x^2}{2}\right) e^{2x}$$

Step 3: Thus the general solution to

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

is

$$y = y_h + y_p$$

$$= c_1 e^{2x} + c_2 x e^{2x} + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)e^{2x}$$

Ex: Let's solve

$$y'' + y = \tan(x)$$

Step 1: First we solve the homogeneous equation $y'' + y = 0$ which has characteristic equation

$$r^2 + 1 = 0$$

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\pm\sqrt{-4}}{2} = \frac{\pm 2\sqrt{-1}}{2}$$

$$= \pm\sqrt{-1} = \pm i$$

$$= 0 \pm 1 \cdot i$$

Thus, the homogeneous general solution is

$$y_h = c_1 e^{0 \cdot x} \cos(1 \cdot x) + c_2 e^{0 \cdot x} \sin(1 \cdot x)$$

$$= c_1 \underbrace{\cos(x)}_{y_1} + c_2 \underbrace{\sin(x)}_{y_2}$$

Step 2: Now we use variation of parameters with $y_1 = \cos(x)$, $y_2 = \sin(x)$.

We have

$$W(y_1, y_2) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) - (-\sin^2(x))$$

$$= 1$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\, dx = \int \frac{-\sin(x) \cdot \tan(x)}{1}\, dx$$

$$= \int \frac{-\sin^2(x)}{\cos(x)}\, dx = \int \frac{(\cos^2(x) - 1)}{\cos(x)}\, dx$$

$$= \int \cos(x)\, dx - \int \sec(x)\, dx$$

$$= \sin(x) - \ln|\sec(x) + \tan(x)|$$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}\, dx = \int \frac{\cos(x) \tan(x)}{1}\, dx$$

$$= \int \cos(x) \cdot \frac{\sin(x)}{\cos(x)}\, dx = \int \sin(x)\, dx = -\cos(x)$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left[\sin(x) - \ln|\sec(x) + \tan(x)|\right] \cdot \cos(x)$$

$$- \cos(x) \cdot \sin(x)$$

$$= -\cos(x) \cdot \ln|\sec(x) + \tan(x)|$$

Step 3: The general solution to $y'' + y = \tan(x)$ is thus

$$y = y_h + y_p$$

$$= c_1 \cdot \cos(x) + c_2 \cdot \sin(x) - \cos(x) \cdot \ln|\sec(x) + \tan(x)|$$

The following contains a theorem that we used above. I put the proof, but its mainly for me or for those that want to see why. It uses some linear algebra results.

Theorem: Let I be an interval. Let $a_1(x)$ and $a_2(x)$ be continuous on I.

Let y_1 and y_2 be linearly independent solutions to $y'' + a_1(x)y' + a_2(x) = 0$.

Then $W(y_1, y_2)(x) \neq 0$ for any x in I.

proof: Let y_1 and y_2 be linearly independent solutions to $y'' + a_1(x)y' + a_0(x)y = 0$.

Suppose that we have x_0 in I with

$$W(y_1, y_2)(x_0) = 0.$$

Then

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$$

would not be invertible.

Thus there would exist constants c_1, c_2 not both zero where

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, we would get

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

Define the function $f(x) = c_1 y_1(x) + c_2 y_2(x)$.
Then by linearity we would have that $f(x)$ would solve $y'' + a_1(x) y' + a_0(x) y = 0$.
Also, $y(x_0) = 0$ and $y'(x_0) = 0$ by the system above.
But the zero function also satisfies $y'' + a_1(x) y' + a_0(x) y = 0$, $y'(x_0) = 0$, $y(x_0) = 0$.
Thus, f is the zero function on I by the uniqueness theorem for second order linear initial value problems.
So, $c_1 y_1(x) + c_2 y_2(x) = 0$ for all x in I.
But then y_1 and y_2 are not linearly independent.
Contradiction.
Thus, $W(y_1, y_2)(x) \neq 0$ for all x in I.

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